CONTRIBUTION TO THE PROBLEM ON THE ANALYTICAL CONSTRUCTION OF REGULATORS

(K ZADACHE OB ANALITICHESKOM KONSTRUIROVANII REGULIATOROV)

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Necessary and sufficient conditions are given for which the stabilization of a linear system is possible under the condition that the integral mean-square error (relative to arbitrary initial disturbances) be a minimum. An explanation is given for the manifold of initial data for which the system can have an optimum stabilization if these conditions are satisfied.

1. Let a control system be described by the equation

$$\frac{dx_k}{dt} = \sum_{j=1}^{n} a_{kj}x_j + b_k u \qquad (k = 1, \ldots, n) \tag{1.1}$$

where the x_k are phase coordinates, a_{kj} and b_k are constant parameters, while u is the guidance action developed within the control device. The system (1.1) can also be written in the matrix form

$$\frac{dx}{dt} = Ax + bu$$

Let us assume that at the initial instant $t_0 = 0$ the coordinates of the system are $x(0) = x_0$. As a criterion for an optimum we shall consider the functional [1]

$$J(u) = \int_0^{\infty} V(u)\,dt, \qquad \left(V(u) = \sum_{k=1}^{n} a_k x_k^2 + cu^2\right) \tag{1.2}$$

Here V is a positive-definite form. It is required to select the control $u(x)$ so that the functional (1.2) will attain the smallest possible value. The solution of this problem, the optimum control $u_0 = u_0(x)$, will make the system (1.1) asymptotically stable, and will insure that the integral error of $J(u_0)$ in deviation from the trajectory $x(t)$ will be a minimum. We are looking for a control not as a function of time t

but as a function of the coordinates of the system $x_1, \ldots, x_n$. This is the problem on the analytical construction of control systems which was considered in the works of Letov [1]. The aim of the present note is to determine the conditions under which the given problem has a solution.

2. Let us consider the question on the existence of an admissible control for the system (1.1).

Definition (2.1). The function $u(t)$ will be said to be an admissible control if $u(t)$ satisfies the inequality $J(u) < +\infty$.

We introduce the following notation. If B is an $n \times n$ matrix and c is an n-dimensional vector, then the symbol $(B_c)_i$ will denote the ith component of their product. Furthermore, let the symbol $(a \cdot b)$ stand for the scalar (inner) product of the vectors a and b.

Let us assume at first that the vectors

$$b,\ Ab,\ \ldots,\ A^{n-1}b \tag{2.1}$$

are linearly independent. In this case one can construct an admissible control for an arbitrary initial condition x_0.

In fact, if x_0 is a fixed point, then for every $t > 0$ there exists a number $N(x_0, t)$ such that

$$\min_{(l \cdot x_0) = -1} \int_0^t (l \cdot F^{-1}(\tau)\, b)^2\, d\tau > \frac{1}{N^2(x_0, t)}, \qquad \sum_{i=1}^n l_i^2 \neq 0$$

Here, $F(\tau)$ is the fundamental matrix of the solution of the system (1.1) when $U \equiv 0$. This means [2, p. 629] that there exists a control $u_1(\tau)$ which transfers the point x_0 into the origin of the coordinate system during the time $\tau = t$, whereby

$$\int_0^t u_1^2(\tau)\, d\tau \leqslant \frac{1}{N^2(x_0, t)}$$

Setting

$$u^*(\tau) = \begin{cases} u_1(\tau) & (0 \leqslant \tau \leqslant t) \\ 0 & (t < \tau) \end{cases}$$

and evaluating $J(u^*)$, we arrive at the conclusion that $u^*(\tau)$ is an admissible control.

Next, let us assume that there are only $k\,(k < n)$ linearly independent vectors among the vectors (2.1). It is not difficult to see that the first k vectors have this property. Let us complete the system

$$b,\ Ab, \ldots, A^{k-1}b \tag{2.2}$$

with vectors $c^{(k+1)}, \ldots, c^{(n)}$ in such a way that the vectors

$$b,\ Ab, \ldots, A^{k-1}b,\ c^{(k+1)}, \ldots, c^{(n)}$$

will form a basis, and let us make the transformation $x = Dy$, where the matrix D has the form

$$D = \left\| \begin{matrix} b_1 & (Ab)_1 & \ldots & (A^{k-1}b)_1 & c_1{}^{(k+1)} & \ldots & c_1{}^{(n)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ b_n & (Ab)_n & \ldots & (A^{k-1}b)_n & c_n{}^{(k+1)} & \ldots & c_n{}^{(n)} \end{matrix} \right\|$$

Since $A^k b$ is expressible linearly in terms of the vectors of the system (2.2)

$$A^k b = \mu_0 b + \mu_1 Ab + \ldots + \mu_{k-1} A^{k-1} b,\ \sum_{i=0}^{k-1} \mu_k{}^2 \neq 0$$

we obtain after some elementary transformations the following set of equations:

$$\frac{dy_1}{dt} = \mu_0 y_k + u, \quad \frac{dy_2}{dt} = y_1 + \mu_1 y_k, \quad \ldots, \frac{dy_k}{dt} = y_{k-1} + \mu_{k-1} y_k \quad (2.3)$$
$$\frac{dy_{k+1}}{dt} = \alpha_{k+1,k+1} y_{k+1} + \cdots + \alpha_{k+1,n} y_n, \cdots, \quad \frac{dy_n}{dt} = \alpha_{n,k+1} y_{k+1} + \cdots + \alpha_{nn} y_n$$

In these equations the elements a_{ij} are given by the formulas

$$\alpha_{ij} = \frac{1}{\Delta} \sum_{m=1}^{n} D_{mi} (Ac^{(j)})_m$$

Here Δ is the determinant of the matrix D, while D_{mi} is the algebraic cofactor of its element d_{mi}.

From Equations (2.3) it follows that the control u acts only on the first k coordinates; the coordinates $y_{k+1}, \ldots, y_n$ are independent of u. Let us introduce into our discussion the matrix

$$A_1 = \left\| \begin{matrix} 0 & 0 & \ldots & \mu_0 \\ 1 & 0 & \ldots & \mu_1 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ 0 & 0 & \ldots\ 1 & \mu_{k-1} \end{matrix} \right\|$$

and let us denote the k-dimensional vector $(1, 0, \ldots, 0)$ by b^*. It is obvious that the vectors

$$b^*, \quad A_1 b^*, \ldots, A_1^{k-1} b^* \tag{2.4}$$

are linearly independent. This means that for every point of the space $\{y_1, \ldots, y_k\}$ there exists an admissible control. The question on the existence of an admissible control for points of the space $\{x_1, \ldots, x_n\}$ is resolved by the properties of the matrix

$$\begin{Vmatrix} \alpha_{k+1,\, k+1} & \cdots & \alpha_{k+1,\, n} \\ \cdots & \cdots & \cdots \\ \alpha_{n,\, k+1} & \cdots & \alpha_{nn} \end{Vmatrix} \tag{2.5}$$

If the roots λ_i of the characteristic equation of the matrix (2.5) satisfy the condition

$$\operatorname{Re} \lambda_i < 0 \qquad (i = 1, \ldots, n-k) \tag{2.6}$$

then there exists an admissible control for every initial state of the system (1.1).

In fact, due to the linear independence of the vectors (2.4), one can construct an admissible control $u^*(\tau)$ for every point $(y_1^\circ, \ldots, y_k^\circ)$ of the space $\{y_1, \ldots, y_k\}$. And since by hypothesis $\lim y_i = 0$ $(i = k+1, \ldots, n)$ as $t \to \infty$, the function $u^*(\tau)$ is an admissible control also for the point $(y_1^\circ, \ldots, y_k^\circ, y_{k+1}, \ldots, y_n)$.

Suppose that only m roots of the characteristic equation of the matrix (2.5) satisfy the condition (2.6). In this case the set of initial data of the system (1.1) for which there exist admissible controls is a space of $k + m$ dimensions.

In fact, the initial values of the asymptotically stable integral curves of Equations (2.3) fill an m-dimensional subspace. Let us denote this subspace by $\{y_s, \ldots, y_{s+m}\}$, $s > k$.

It is not difficult to see that the direct sum of the subspaces

$$\{y_1, \ldots, y_k\} \text{ and } \{y_s, \ldots, y_{s+m}\}$$

consists of those points for which one can construct admissible controls $u^*(\tau)$. Just as above, the function $u^*(\tau)$ will be an admissible control point $(y_1, \ldots, y_k)$.

The converse to these statements is also valid.

If an admissible control exists for every initial state of the system (1.1), then the vecotrs (2.1) are either linearly independent, or for some $k < n$ the vectors (2.2) are linearly independent and the matrix

(2.5) satisfies the condition (2.6).

If the set of initial data for which one can construct admissible controls is a subspace of the kth dimension, then the vectors (2.2) are either linearly independent and the roots of the characteristic equation of the matrix (2.5) satisfy the inequality $\operatorname{Re}\lambda_i \geqslant 0$ ($i = 1, \ldots, n-k$), or among the vectors (2.2) there are only k_1 linearly independent ones ($k_1 < k$), and ($k - k_1$) of the roots of the characteristic equation of the matrix (2.5) have the property (2.6).

The validity of these assertions follows from earlier considerations.

When an admissible control $u^*(\tau)$ as a function of time has been constructed, then one can assert that there exists an admissible control as a function of the system's coordinates, i.e. $u^*(\tau) = u(x(\tau))$. We note here that if the admissible control $u(x)$ exists for the initial condition $x(0) = x_0$, then an admissible control exists also for some region of the initial data; namely, it exists for the points $x(x_0, u^*, t)$ of the trajectory of the system (1.1) where we have set $u^* = u^*(\tau)$, $t > 0$.

3. Let us prove the following theorem.

Theorem 3.1. If there exists an admissible control $u(x)$ for the region G_0 of initial data, then there exists also an optimum control $u_0(x)$ for the region G_0, and we have

$$u_0(x) = p_1x_1 + \ldots + p_nx_n \tag{3.1}$$

Here the p_i are constants determined by the functional and the parameters of the system under consideration.

Proof. We shall make use of the results of [3] (p. 248). In accordance with these results, the minimizing of the functional (1.2) leads to the solution of the following variational problem. It is required to determine

$$\min_u \left[\sum_{k=1}^{n} \frac{\partial M(x)}{\partial x_k}\frac{dx_k}{dt} + V(u)\right] = 0 \qquad (M(x_0) = \min_u J(u),\ x_0 \in G_0) \tag{3.2}$$

where the optimum control u_0 is the solution of the problem (3.2).

Hence, the optimum control satisfies the equations

$$\sum_{k=1}^{n} \left[\frac{\partial M(x)}{\partial x_k}\Big(\sum_{j=1}^{n} a_{kj}x_j + b_ku\Big) + a_kx_k^2\right] + cu^2 = 0, \qquad \sum_{k=1}^{n} \frac{\partial M(x)}{\partial x_k} b_k + 2cu = 0 \tag{3.3}$$

In the case under consideration, $V(u)$ is a positive-definite form. Therefore, if it is possible to find a positive-definite quadratic form

$M(x)$ and a control

$$u_0 = -\frac{1}{2c}\sum_{k=1}^{n}\frac{\partial M(x)}{\partial x_k}b_k = \sum_{k=1}^{n} p_i x_i$$

which are a solution of the problem (3.2), then

$$[M(x)]_{t=0} = \min_u J(u) = J(u_0)$$

Indeed, by (3.3) we have

$$\frac{dM(x)}{dt} = -V(u) \tag{3.4}$$

This means that for the system (1.1) there exists a positive-definite form $M(x)$ whose total derivative with respect to time is by (1.1) a negative-definite function.

According to Liapunov's theorem [4, p. 32] the system (1.1) is asymptotically stable; hence $M(x) = 0$ when $t = +\infty$.

Therefore, through integration of (3.4) with respect to t from 0 to $+\infty$, we obtain

$$[M(x)]_{t=0} = \int_0^{\infty} V(u_0)\,dt$$

Let us assume that for some initial condition x_0 the minimum of the functional (1.2) is obtained for the control $u^*(x) \neq u_0(x)$, i.e.

$$J(u_0) > J(u^*) \tag{3.5}$$

Then the next inequality will hold:

$$\frac{dM(x)}{dt} + V(u^*) \geqslant 0$$

Therefore, $[M(x)]_{t=0} = J(u_0) \leqslant J(u^*)$, which contradicts (3.5).

This means that the existence of an optimum control will have been proved if we show that there exists a positive-definite quadratic form $M(x)$ and a linear function $u = p_1x_1 + \ldots + p_nx_n$ which satisfies condition (3.2). Let us introduce the auxiliary system

$$\frac{dx_k}{dt} = \theta\sum_{j=1}^{n} a_{kj}x_j + \theta b_k\xi + (1-\theta)u_k - (1-\theta)x_k \qquad (k=1,\ldots,n) \tag{3.6}$$

where θ is a positive parameter, $0 \leqslant \theta \leqslant 1$.

Suppose that it is required to select the $u_1, \ldots, u_n, \xi$ so that the functional

$$J^{(1)}(u,\xi) = \int_0^\infty V^{(1)}(u,\xi)\,dt = \int_0^\infty \Big[\sum_{k=1}^n (a_k x_k^2 + (1-\theta)u_k^2) + \theta c\xi^2\Big]\,dt$$

might take on its smallest value.

The system (3.6) has an admissible control for every value of the parameter $\theta \geqslant 0$ and for every initial state if an admissible control exists for the original system (1.1). When $\theta = 1$, the system (3.6) is transformed into the system (1.1).

Suppose that $x_0 \in G_0$. We shall show that an optimum control exists for Equation (3.6) when θ has any value in the interval $0 \leqslant \theta \leqslant 1$.

Indeed, when $\theta = 0$, it follows from (3.6) that

$$\frac{dx_k}{dt} = -x_k + u_k \qquad (k = 1, \ldots, n) \tag{3.7}$$

Since the control u_k acts only on the coordinate x_k, we shall, in place of minimizing the functional $J^{(1)}(u, \xi)$, try to find the minimum of the expression

$$\int_0^\infty (a_k x_k^2 + u_k^2)\,dt$$

for each k separately.

From Equations (3.3) we have

$$a_k x_k^2 + u_k^2 + \frac{\partial M(x)}{\partial x_k}(-x_k + u_k) = 0, \qquad 2u_k + \frac{\partial M(x)}{\partial x_k} = 0$$

Whence

$$\left(\frac{\partial M(x)}{\partial x_k}\right)^2 + 4\left(\frac{\partial M(x)}{\partial x_k}\right)x_k - 4a_k x_k^2 = 0$$

This means that

$$u_k = (1 \pm \sqrt{1 + a_k})\,x_k \quad .$$

Since the optimum control u_k° must make the system (3.7) asymptotically stable, we have

$$u_k^\circ = (1 - \sqrt{1 + a_k})\,x_k \qquad (k = 1, \ldots, n) \tag{3.8}$$

Let us write Equations (3.3) for the system (3.6)

$$\sum_{k=1}^{n} \frac{\partial M^{(1)}}{\partial x_k}\Big[\theta \sum_{j=1}^{n} a_{kj}x_j + \theta b_k \xi - (1-\theta)\,x_k + (1-\theta)\,u_k\Big] + V^{(1)}(u, \xi) = 0$$

$$2u_k + \frac{\partial M^{(1)}}{\partial x_k} = 0 \qquad (k = 1, \ldots, n), \qquad 2\xi c + \sum_{k=1}^{n} \frac{\partial M^{(1)}}{\partial x_k} b_k = 0$$

From this we obtain the next equations for the determination of the function $M^{(1)}(x)$:

$$\sum_{k=1}^{n} \frac{\partial M^{(1)}}{\partial x_k}\Big[\theta \sum_{j=1}^{n} a_{kj}x_j + \theta b_k \xi - (1-\theta)\,x_k + (1-\theta)\,u_k + \\ + \sum_{k=1}^{n} a_k x_k^2 + \frac{\theta}{4}\Big(\sum_{k=1}^{n} \frac{\partial M^{(1)}}{\partial x_k} b_k\Big)^2 + \frac{1}{4} \sum_{k=1}^{n} \Big(\frac{\partial M^{(1)}}{\partial x_k}\Big)^2 = 0 \qquad (3.9)$$

Let us assume that the solution of the problem (3.2) yields some positive-definite form

$$M^{(1)}(\theta, x) = \sum_{i,\, j=1}^{n} b_{ij}(\theta)\, x_i x_j, \qquad (b_{ij} = b_{ji}) \qquad (0 \leqslant \theta \leqslant 1)$$

and the controls

$$u_k^\circ(\theta, x) = -\frac{1}{2}\frac{\partial M^{(1)}(\theta, x)}{\partial x_k} = \sum_{i=1}^{n} p_i^{(k)}(\theta)\, x_i$$

$$\xi^\circ(\theta, x) = -\frac{1}{2c}\sum_{k=1}^{n} \frac{\partial M^{(1)}(\theta, x)}{\partial x_k} b_k = \sum_{k=1}^{k} p_i(\theta)\, x_i \qquad (3.10)$$

where $p_i^{(k)}(\theta)$ and $p_i(\theta)$ are constants for fixed θ.

Differentiating (for the time being, just formally) the expression (3.9) with respect to θ, we obtain

$$\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\frac{\partial M^{(1)}}{\partial \theta}\Big[\theta \sum_{j=1}^{n} a_{kj}x_j + \theta \xi b_k - (1-\theta)\,x_k + (1-\theta)\,u_k\Big] \qquad (3.11)$$

$$= -\sum_{k=1}^{n} \frac{\partial M^{(1)}}{\partial x_k}\Big(\sum_{j=1}^{n} a_{kj}x_j + x_k\Big) - \frac{1}{4}\Big[\Big(\sum_{k=1}^{n} \frac{\partial M^{(1)}}{\partial x_k} b_k\Big)^2 - \sum_{k=1}^{n}\Big(\frac{\partial M^{(1)}}{\partial x_k}\Big)^2\Big]$$

The validity of the operation of differentiation can be established if it is possible to determine the coefficients of the form $\partial M^{(1)}(\theta,x)/\partial\theta$ by means of (3.11). In order to accomplish this one has to substitute into (3.11) the form $M^{(1)}(\theta, x)$ and equate the coefficients of like powers of x_i and $x_i x_j$. Let us show that in this manner we can actually obtain equations for the coefficients of the form $\partial M^{(1)}(\theta, x)/\partial\theta$ which have a unique solution $db_{ij}(\theta)/dt$. Let us denote by $W(\theta)$ the Jacobian of the system of equations for the determination of the $db_{ij}(\theta)/d\theta$ obtained from (3.1I).

If for some value $\theta \geqslant 0$ the problem (3.2) has for its solution a form $M^{(1)}(\theta, x)$, then the right-hand side of the relation (3.11) will be some quadratic form. The left-hand side of (3.11) is the total derivative of the function $\partial M^{(1)}(\theta, x)/\partial\theta$, evaluated on the basis of (3.6) for the value of θ under consideration. Since $M^{(1)}(\theta, x)$ is a positive-definite form, and since $dM^{(1)}(\theta, x)/dt$ is a negative-definite quadratic form for the controls (3.10) as a consequence of (3.9), the system (3.6) will be asymptotically stable for the given θ. Therefore [4, p. 61] the coefficients $db_{ij}(\theta)/d\theta$ of $\partial M^{(1)}(\theta, x)/\partial\theta$ can be uniquely determined by means of (3.11) as some functions of the $b_{ij}(\theta)$, $i, j = 1, \ldots, n$, and of the parameter θ:

$$\frac{db_{ij}(\theta)}{d\theta} = \Phi_{ij}(b_{s,e}(\theta), \theta), \qquad (s, e = 1, \ldots, n) \tag{3.12}$$

Starting with the relation (3.11) one can show that the functions Φ_{ij} depend continuously on the b_{ij} and on θ for all those values of b_{ij} and θ for which the Jacobian $W(\theta)$ is distinct from zero. This makes it possible to determine the coefficients $b_{ij}(\theta)$ for all values of the parameter in the interval $0 \leqslant \theta \leqslant 1$. In accordance with what has been said above, it is sufficient for this purpose to show that the Jacobian $W(\theta)$ is different from zero, $0 \leqslant \theta \leqslant 1$, and that for no values of i, j and θ_1 can the following relations hold:

$$\lim b_{ij}(\theta) = \infty \quad \text{when } \theta \to \theta_1 - 0 \tag{3.13}$$

Let us consider the solution of the system (3.12) with the initial condition $\theta = 0$, $b_{ij}(0)$ (the coefficients $b_{ij}(0)$ of $M^{(1)}(0, x)$ are completely determined by Formulas (3.8)). This solution exists at least in a small enough neighborhood $0 \leqslant \theta \leqslant \mu$ of the point $\theta = 0$. It is our problem to show that this solution can be extended to all θ in the interval $0 \leqslant \theta \leqslant 1$.

Let us select an admissible control $u(\theta, x)$, $\xi(\theta, x)$ so that the following inequality be satisfied

$$J^{(1)}(u(\theta, x), \xi(\theta, x)) \leqslant E, \qquad (0 \leqslant \theta \leqslant 1) \tag{3.14}$$

where $E = \text{const} > 0$. Such a choice of an admissible control is possible on the basis of the results of Section 2 of the present article and by the hypothesis of the theorem.

Let us suppose that the solution of Equation (3.12) is extendible only to the neighborhood $0 \leqslant \theta < \theta_1 < 1$ of the point $\theta = 0$. This can happen in two cases. Namely, if for $\theta \to \theta_1 - 0$ the system (3.6) loses its asymptotic stability, and hence it is impossible to determine the coefficients in $\partial M^{(1)}(\theta, x)/\partial\theta$ in the form (3.12) when $\theta = \theta_1$, or if for $\theta \to \theta_1 - 0$ Equations (3.13) apply.

Let us consider the first case. The form $M^{(1)}(\theta, x)$ must lose its character of being positive-definite when $\theta \to \theta_1 - 0$. This is impossible, for $M^{(1)}(\theta, x)$ gives a minimum for (1.2). This means the first case cannot occur. If, however, Equations (3.13) apply, then for values of θ near enough to θ_1 we would have

$$J^{(1)}(u^\circ(\theta, x),\ \xi^\circ(\theta, x)) > E$$

which is impossible because of the fact that the admissible control $u(\theta, x)$ satisfies the inequality (3.14). This completes the proof of the theorem.

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